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HEAT AND MASS TRANSFER AND PHYSICAL GASDYNAMICS

Turbulent Boundary Layer under Simultaneous Effect of the Longitudinal Pressure Gradient, Injection (Suction), and Transverse Surface Curvature

A. V. Garbaruk, Yu. V. Lapin, and M. Kh. Strelets

St. Petersburg State Technical University, St. Petersburg, 195251 Russia

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Abstract—The procedure successfully used in the construction of an algebraic model of turbulence for flows with an adverse pressure gradient [1] is used for the construction of a more general model of turbulence allowing for both the direct and cross effect of the factors acting simultaneously, namely, the longitudinal pressure drop, injection (suction) of a gas through a porous surface, and the transverse surface curvature. In contrast to the model proposed in [2] and based on the use of the Bradshow–Ferriss–Atwell equation for the turbulent stresses in determining the velocity scale in the outer region, the model suggested by us is fully based on the equation for the first moments written in terms of the law of the wall. The latter fact makes it possible to use only two empirical constants, as in [1], and to eliminate the stage of "tuning" the model, as in [2], which is associated with the fitting of three additional empirical constants characterizing the diffusion of shear stress under the effect of three external factors, namely, the pressure drop, injection (suction), and transverse curvature. In so doing, the testing results indicate that the proposed model proves to be quite competitive both with the model of [2] and with the most representative differential models of turbulence.

FORMULATION OF THE PROBLEM AND GENERAL FORMULATION OF THE MODEL

The equations of two-dimensional (plane and axisymmetric) stationary boundary layer have the form

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{dp}{dx} + \frac{1}{r^{\alpha}}\frac{\partial}{\partial y}(r^{\alpha}\tau), \quad (1)$$

$$\frac{\partial(r^{\alpha}u)}{\partial x} + \frac{\partial(r^{\alpha}v)}{\partial y} = 0. \quad (2)$$

Here, u, v and x, y are the velocity projections and the coordinate axes, directed along the surface subjected to flow and on the normal to this surface, respectively; p is the pressure; τ is the shear stress; $r = r_w + Ky$ is the local radius of transverse curvature ($K = +1$ for a convex surface and –1 for a concave one); r_w is the radius of transverse surface curvature; $\alpha = 0$ denotes the plane flow; and $\alpha = 1$ denotes the axisymmetric flow.

The boundary conditions for the set of equations (1) and (2) are written as

$$u = 0,\ v = V_w \ \text{at}\ y = 0;\ u \longrightarrow U_e \ \text{at}\ y \longrightarrow \infty, \quad (3)$$

where V_w is the rate of injection ($V_w > 0$) or suction ($V_w < 0$) and U_e is the velocity on the outer boundary of the boundary layer.

We preserve the linear scales (y, δ^* for the inner i and outer o regions, respectively) that are traditional for the two-layer Klauser scheme of the boundary layer and do not touch upon the problem of determination of the velocity scales V_{si} and V_{so} for the same regions to write the model of turbulent viscosity in the following form:

$$\nu_T = \kappa\min\{yV_{si}D,\ \delta^*V_{so}\gamma\}. \quad (4)$$

We adopt the same damping factor D and intermittency factor γ as in [1],

$$D = [1 - \exp(-yV_{si}/\nu A)]^3,$$
$$\gamma = [1 + 5.5(y/\delta)^6]^{-1}, \quad \kappa = 0.436, \quad A = 13. \quad (5)$$

Here, ν is the kinematic viscosity, δ is the thickness of the boundary layer, and κ is the Karman constant.

For axisymmetric flows, the linear scale of the inner region, which is provided by the displacement thickness of the boundary layer δ^*, is defined by the relation [3]

$$\delta^* = \sqrt{r_w^2 + 2\int_0^{\delta}(1 - u/U_e)(r_w + y)dy} - r_w. \quad (6)$$

For a plane flow ($r_w \longrightarrow \infty$), expression (6) assumes the conventional form

$$\delta^* = \int_0^{\delta}(1 - u/U_e)dy. \quad (7)$$

For flows in which the shear stress τ varies monotonically from the maximum value on the wall τ_w to zero on the outer boundary, i.e., which satisfy the condition $(\partial(\tau)/\partial y)_{y=0} \le 0$, the available computational experience [1, 2, 4] is indicative of the fact that one and the same scale, i.e., the dynamic velocity ($V_* = \sqrt{\tau_w/\rho}$), may be used as the velocity scales V_{si} and V_{so}: $V_{si} = V_{so} = V_*$.

In flows with a nonmonotone pattern of variation of the shear stress with a maximum of τ within the layer, i.e., which satisfy the condition $(\partial(\tau)/\partial y)_{y=0} > 0$, the quantity $V_{si} = \sqrt{\tau/\rho}$ may be adopted as the scale V_{si}. As for V_{so}, in the plane case, we will determine it by the maximum value of τ_m: $V_{so} = \sqrt{\tau_m/\rho}$. The position of the point of maximum of τ and the value of τ_m may be obtained directly from the equation of motion with due regard for the fact that $\partial(\tau)/\partial y)_{y=y_m} = 0$ [1, 2]. The method of determining V_{so} in the axisymmetric case will be discussed in detail below.

THE VELOCITY SCALE IN THE INNER REGION V_{si}

To determine the velocity scale in the inner region, we use Eqs. (1) and (2) written in variables of the law of the wall [5]

$$\xi = x,\ \eta = yV_*/\nu,\ \varphi = \varphi(\eta),\ \varphi = u/V_*. \quad (8)$$

We eliminate the transverse velocity v from Eq. (1) using continuity equation (2) to obtain

$$\frac{1}{r^+}\frac{\partial(r^+\bar{\tau})}{\partial\eta} = \frac{\nu}{V_*^2}\frac{dV_*}{dx}\varphi^2\left(1 + \frac{K}{r^+\varphi^2}\frac{d\varphi}{d\eta}\int_0^\eta \varphi\eta\, d\eta\right) + \frac{\nu}{V_*\tau_w}\frac{dp}{dx} + B_*\frac{r_w}{r}\frac{d\varphi}{d\eta} + \frac{1}{r^+}\frac{dr_w}{dx}\frac{d\varphi}{d\eta}\int_0^\eta \varphi\, d\eta. \quad (9)$$

In Eq. (9), $\bar{\tau} = \tau/\tau_w$, $B_* = V_w/V_*$ is the injection (suction) parameter and $r^+ = rV_*/\nu$.

We assume that the convection terms (first term on the right-hand side of (9)) in the inner region may be ignored and confine the treatment to the particular case r_w = const to derive

$$\frac{1}{r^+}\frac{\partial(r^+\bar{\tau})}{\partial\eta} = \frac{\nu}{V_*\tau_w}\frac{dp}{dx} + B_*\frac{r_w}{r}\frac{d\varphi}{d\eta}. \quad (10)$$

The integration of this equation from zero to η yields the expression for the distribution of shear stress in the inner region of the boundary layer,

$$\bar{\tau} = \frac{r_w}{r}\left[1 + B_*\varphi + \frac{dp}{dx}\frac{y}{\tau_w}\left(1 + \frac{K}{2}\frac{y}{r_w}\right)\right]. \quad (11)$$

With due regard for the previously postulated correlation between τ and V_{si} ($V_{si} = \sqrt{\tau/\rho}$) and relation (11) for the shear stress, the expression for the velocity scale in the inner region assumes the following form:

$$V_{si} = V_*\sqrt{\frac{1}{1 + Ky/r_w}\left[1 + B_*\varphi + \frac{dp}{dx}\frac{y}{\tau_w}\left(1 + \frac{K}{2}\frac{y}{r_w}\right)\right]}. \quad (12)$$

VELOCITY SCALE IN THE OUTER REGION V_{so}

To determine the velocity scale V_{so} for a plane flow, we assume that the boundary between the inner and outer regions (y_m) coincides with the point of maximum of the shear stress (τ_m), i.e., with the point at which the condition $\partial\tau/\partial y)_{y=y_m} = 0$ is valid. The boundary between the regions is treated as a line on which the balance (local equilibrium) of force effects of all factors acting on the flow including the convection, injection (suction), and the pressure gradient is attained.

In contrast to the plane flow, in the axisymmetric flow, in connection with the emergence of an additional linear scale (r_w), the equation of motion expresses the balance of moments of all forces acting on the flow. Therefore, the boundary (y_m) between the inner and outer regions may be provided by the point at which the maximum of the moment of shear stress ($\tau_m r_m$) is attained, i.e., the point at which the condition $((\partial(r\tau)/\partial y)_{y=y_m} = 0)$ is valid. In this case, the velocity scale will be defined by the value of τ_m that takes place at the point of maximal moment of shear stress rather than at the point corresponding to the maximal value of $\tau = \tau_{\max}$.

In both cases, to determine both the maximal shear stress τ_m in a plane flow and τ_m at the point of the maximal moment of shear stress in an axisymmetric flow, one can use the integral of equation of motion (9). This equation written in variables of the law of the wall is valid in the entire inner region including its boundary with the outer region defined as described above. However, before the integration of this equation, we will perform some simplifications. In particular, it can be demonstrated that the effect of transverse curvature on the convection, expressed by the second summand in the first term on the right-hand side of Eq. (9), proves to be minor in the region of the logarithmic velocity profile regardless of the value of parameter δ/r_w. In addition, as previously, the treatment will be confined to the case of r_w = const, $dr_w/dx = 0$.

With due regard for the estimates made and limitations, Eq. (9) may be written in the following form:

$$\frac{\partial\bar{\tau}}{\partial\eta} = \frac{\nu}{V_*^2}\frac{dV_*}{dx}\varphi^2 + \frac{\nu}{V_*\tau_w}\frac{dp}{dx} + B_*\frac{r_w}{r}\frac{d\varphi}{d\eta} - K\frac{\bar{\tau}}{r^+}. \quad (13)$$

We integrate this equation from the wall to the boundary between the inner and outer regions ($0 \le \eta \le \eta_m$) to obtain

$$\bar{\tau}_m = 1 + \frac{\nu}{V_*^2}\frac{dV_*}{dx}\int_0^{\eta_m}\varphi^2 d\eta + \frac{dp}{dx}\frac{\nu\eta_m}{V_*\tau_w} + B_*\int_0^{\eta_m}\frac{r_w}{r}\frac{d\varphi}{d\eta}d\eta - K\int_0^{\eta_m}\frac{\bar{\tau}}{r^+}d\eta. \quad (14)$$

The condition of existence of a maximum of the shear stress within the boundary layer,

$$\left(\frac{\partial\bar{\tau}}{\partial\eta}\right)_{\eta=0} = \frac{\nu}{V_*\tau_w}\frac{dp}{dx} + B_*\left(\frac{d\varphi}{d\eta}\right)_{\eta=0} - \frac{K}{r_w^+} > 0, \quad (15)$$

follows directly from Eq. (13) and boundary conditions on the wall (3).

For a plane flow ($r_w \longrightarrow \infty$), this condition is simplified in view of the vanishing of the last term on the right-hand side of equality (15). If condition (15) is valid, i.e., with a certain correlation between the "external" factors (longitudinal pressure drop, injection (suction), transverse curvature), a local equilibrium (balance) of moments of all forces acting on the flow is established on the boundary between the inner and outer regions. The condition which expresses this balance follows directly from Eq. (9) with due regard for the vanishing of the derivative of the moment of shear stress: $(\partial(r\tau)/\partial\eta)_{\eta=\eta_m} = 0$. Given the previous assumptions of the possibility of ignoring the effect of transverse curvature on the convection and r_w = const, the condition of local equilibrium takes the form

$$-\frac{\nu}{V_*^2}\frac{dV_*}{dx} = \frac{1}{\varphi_m^2}\left[\frac{\nu}{V_*\tau_w}\frac{dp}{dx} + B_*\frac{r_w}{r_m}\left(\frac{d\varphi}{d\eta}\right)_m\right]. \quad (16)$$

One can easily obtain an analog of this condition for a plane flow if one notes that the factor r_w/r_m in the second term on the right-hand side of Eq. (16) goes to unity at $r_w \longrightarrow \infty$. We eliminate the parameter $\frac{\nu}{V_*^2}\frac{dV_*}{dx}$ from expression (14) using condition (16) to obtain

$$\bar{\tau}_m = 1 + \frac{dp}{dx}\frac{\nu\eta_m}{V_*\tau_w} - \frac{1}{\varphi_m^2}\left[\frac{\nu}{V_*\tau_w}\frac{dp}{dx} + B_*\frac{r_w}{r_m}\left(\frac{d\varphi}{d\eta}\right)_m\right]\int_0^{\eta_m}\varphi^2 d\eta + B_*\int_0^{\eta_m}\frac{r_w}{r}\frac{d\varphi}{d\eta}d\eta - K\int_0^{\eta_m}\frac{\bar{\tau}}{r^+}d\eta. \quad (17)$$

In inverting the integrals appearing on the right-hand side of relation (17), we use their representation in the form of decreasing asymptotic expansions obtained by repeated integration by parts. We confine ourselves to two-term representations, which provide for the necessary accuracy in the calculation of integrals, to obtain

$$\int_0^{\eta_m}\varphi^2 d\eta = \varphi_m^2\eta_m(1-2\Phi_m), \quad \Phi_m = \left(\frac{d\varphi}{d\eta}\right)_m\frac{\eta_m}{\varphi_m}, \quad (18)$$

$$\int_0^{\eta_m}\frac{r_w}{r}\frac{d\varphi}{d\eta}d\eta = \varphi_m\Phi_m\frac{r_w}{r_m}\left[1 + \frac{1}{2}\left(1 + \frac{y_m}{r_m}\right)\right], \quad (19)$$

$$\int_0^{\eta_m}\frac{\bar{\tau}}{r^+}d\eta = \frac{\bar{\tau}_m}{r_m^+}\eta_m\left[1 + \frac{1}{2}\frac{y_m}{r_m}\right]. \quad (20)$$

We use the distribution of $\bar{\tau}$ in the inner region given by Eq. (11) for the detailing of $\bar{\tau}_m$ on the right-hand side of Eq. (20) to derive

$$\int_0^{\eta_m}\frac{\bar{\tau}}{r^+}d\eta = \frac{y_m r_w}{r_m^2}\left(1 + \frac{1}{2}\frac{y_m}{r_m}\right) \times\left[1 + B_*\varphi_m + \frac{dp}{dx}\frac{y_m}{\tau_w}\left(1 + \frac{K}{2}\frac{y_m}{r_w}\right)\right]. \quad (21)$$

The relations for the parameter $\bar{\tau}_m$ (17) and representations of integrals (18) and (19) appearing in this relation include the parameter $d\varphi/d\eta = f(\eta)$, which characterizes the form of the law of the wall in one or another flow. The available experimental data on the velocity profiles in boundary layers are indicative of the fact that the generalized law of the wall [2, 6–9] may be written in the form

$$\frac{d\varphi}{d\eta} = \frac{1}{\kappa\eta}\sqrt{\frac{r_w}{r}(1 + B_*\varphi_m)}. \quad (22)$$

We return to relation (17) to substitute equalities (18), (19), and (21) into it. Then, after necessary simplifications and with due regard for the fact that $V_{so} = \sqrt{\tau_m/\rho}$, we obtain the expression for the velocity scale in the inner region,

$$V_{so} = V_*\left\{1 + 2\Phi_m\frac{dp}{dx}\frac{y_m}{\tau_w} + \frac{1}{2}B_*\varphi_m\Phi_m\frac{r_w}{r_m}\left(1 + \frac{y_m}{r_m} + 4\Phi_m\right) - K\frac{y_m r_w}{r_m^2}\left(1 + \frac{1}{2}\frac{y_m}{r_m}\right)\left[1 + B_*\varphi_m + \frac{dp}{dx}\frac{y_m}{\tau_w}\left(1 + \frac{K}{2}\frac{y_m}{r_w}\right)\right]\right\}^{1/2}, \quad (23)$$

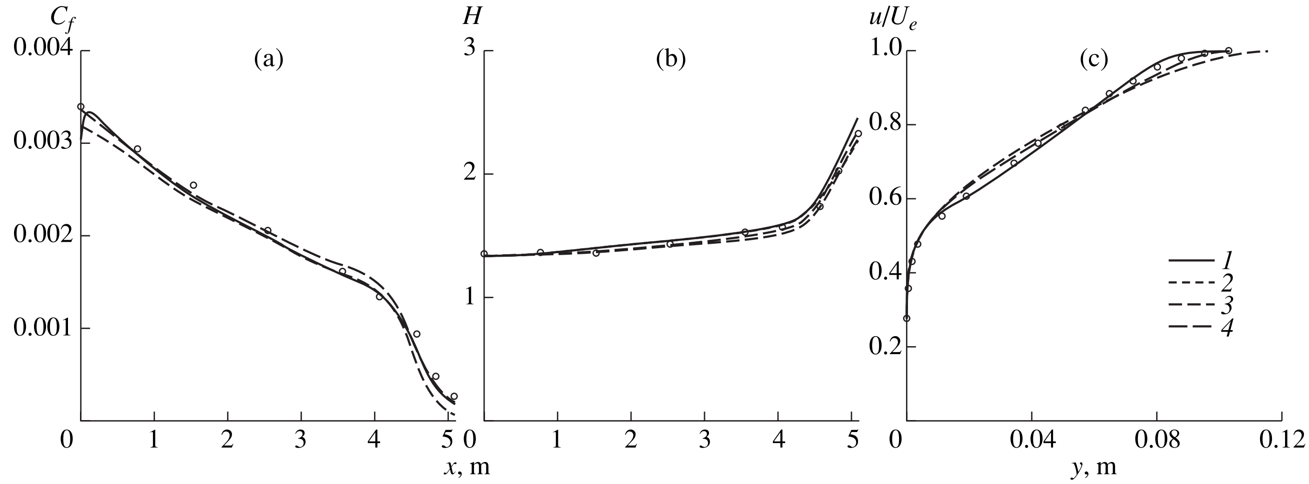

Fig. 1. Longitudinal distributions of the basic parameters of the boundary layer and the velocity profile for experiment 4800 [10]: (a) friction coefficient C_f, (b) form factor H, and (c) velocity profile at $x = 3.556$ m. Calculation (*1*) by our model (relations (4)–(7), (12), (23), (24)), (*2*) by the algebraic model of [2], (*3*) by the Spalart–Allmaras model [13], and (*4*) by the Menter model [14]; dots indicate the experimental data of [10].

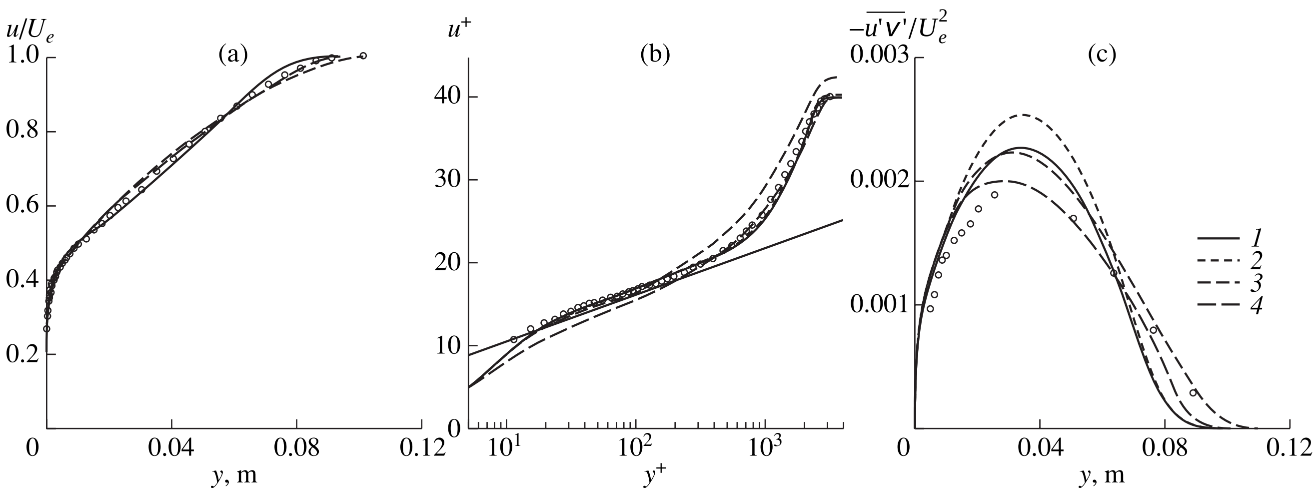

Fig. 2. Calculated and experimental velocity profiles and shear stresses for experiment 0141 [11] at $x = 3.4$ m: (a) velocity profile in physical variables, (b) velocity profile in variables of the law of the wall, and (c) profile of shear turbulent stress $\overline{u'v'}$. Notation is the same as in Fig. 1.

$$\Phi_m = \frac{1}{\kappa\varphi_m}\sqrt{\frac{1 + B_*\varphi_m}{r_m/r_w}}, \quad r_m = r_w + y_m. \qquad (24)$$

The set of relations (4)–(7), (12), and (23), (24) forms the suggested algebraic model of turbulent viscosity for wall boundary layers with the maximum of shear stresses within the layer. Note that the range of validity of the obtained model of turbulence is limited by conditions (15) and (16).

We will further treat some particular cases and confine ourselves to writing only the velocity scale in the outer region V_{so}.

(1) The flow on a plane ($r_w \longrightarrow \infty$) impenetrable ($B_* = 0$) surface in the presence of an adverse pressure gradient ($dp/dx > 0$):

$$V_{so} = V_*\left(1 + \frac{2}{\kappa\varphi_m}\frac{dp}{dx}\frac{y_m}{\tau_w}\right)^{1/2}. \qquad (25)$$

This model was first proposed in [1]. In the latter paper, the results of its fairly detailed testing are also given, which are indicative of its high efficiency in describing the characteristics of turbulent boundary layers with separation.

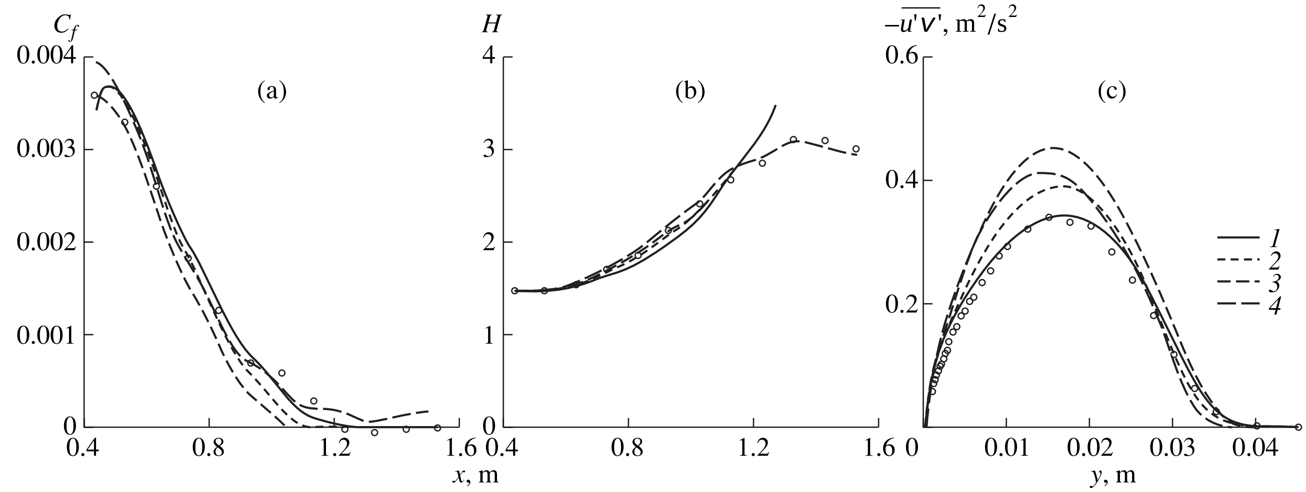

Fig. 3. Longitudinal distributions of the basic parameters of the boundary layer and the profile of shear turbulent stress for experiment DF [3]: (a) friction coefficient C_f, (b) form factor H, and (c) shear stress at $x = 0.931$ m. Notation is the same as in Fig. 1.

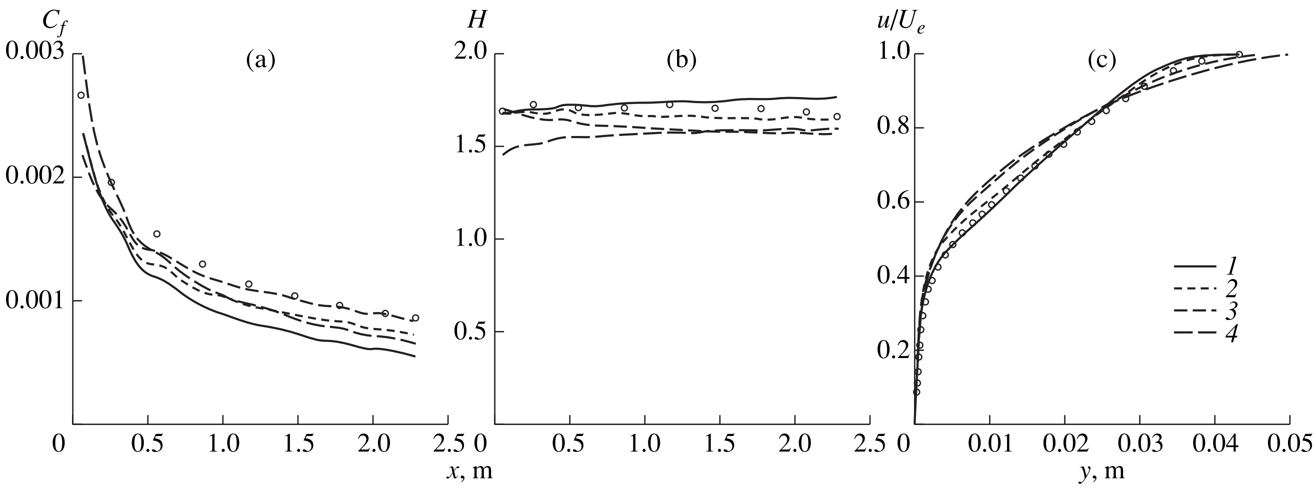

Fig. 4. Longitudinal distributions of the basic parameters of the boundary layer and the velocity profile for experiment 0241 [12]: (a) friction coefficient C_f, and (b) form factor H, (c) velocity profile at $x = 1.168$ m. Notation is the same as in Fig. 1.

(2) The flow on a plane ($r_w \longrightarrow \infty$) penetrable ($B_* > 0$) plate ($dp/dx > 0$), i.e., the model of “pure” injection:

$$V_{so} = V_*\left[1 + \frac{B_*\sqrt{1 + B_*\varphi_m}}{2\kappa}\left(1 + \frac{4\sqrt{1 + B_*\varphi_m}}{\kappa\varphi_m}\right)\right]^{1/2}. \quad (26)$$

(3) The flow on a cylindrical concave surface ($K = -1$):

$$V_{so} = V_*\left[1 + \frac{y_m r_w}{r_m^2}\left(1 + \frac{y_m}{2r_m}\right)\right]^{1/2}. \quad (27)$$

RESULTS OF TESTING THE MODEL

The fact that the experimental data are limited, in particular, their absence for flows subject to a simultaneous effect of all factors treated above (pressure gradient, injection (suction), transverse curvature), complicates considerably full-scale testing of the suggested model. In view of this, in order to test the model, calculations were performed for turbulent boundary layers for which reliable experimental data are available in the literature. Boundary layers on a flat surface were treated in the presence of a moderate (experiment 4800 [10]) and high (experiment 0141 [11]) adverse pressure gradient, as well as a boundary layer with a high adverse pressure gradient on a cylinder subjected to longitudinal flow [3] and on a flat penetrable surface (in the presence of injection) [12]. In addition, the results of calculations using the suggested model were compared with the results obtained on the basis of other, most representative models: the algebraic model of [2], the Spalart and Allmaras differential model with a single equation for turbulent viscosity [13], and the Menter differential k–ω model [14]. Note that the so-called reverse method was used [15] in the calculations of boundary layers with a pressure gradient.

The calculation results are given in figures in the form of longitudinal distributions of the friction coefficient $C_f = 2\tau_w/\rho U_e^2$, the form factor of the boundary layer $H = \delta^*/\theta$ (θ is the momentum thickness), the velocity profiles $u/U_e = f(y)$ and $u^+(y^+)$, and the shear stress $-\overline{u'v'}/U_e^2 = f(y)$.

Analysis of the results given in Figs. 1–4, as well as of the testing results in studies [1, 4], makes it possible to conclude that, for the treated class of flows (boundary layer), the efficiency of the suggested algebraic model is not inferior to that of much more complicated present-day differential models.

REFERENCES

1. Garbaruk, A.V., Lapin, Yu.V., and Strelets, M.Kh., *Teplofiz. Vys. Temp.*, 1999, vol. 37, no. 1, p. 87 (*High Temp.* (Engl. transl.), vol. 37, no. 1, p. 82).
2. Garbaruk, A.V., Lapin, Yu.V., and Strelets, M.Kh., *Teplofiz. Vys. Temp.*, 2001, vol. 39, no. 4, p. 589 (*High Temp.* (Engl. transl.), vol. 39, no. 4, p. 548).
3. Dengel, P. and Fernholz, H.H., *J. Fluid Mech.*, 1990, vol. 212, p. 615.
4. Garbaruk, A.V., Modern Semiempirical Models of Turbulence for Boundary Flows: Testing and Comparison, *Cand. Sci. (Phys.–Math.) Dissertation*, St. Petersburg: St. Petersburg State Technical Univ., 1999.
5. Lapin, Yu.V., *Turbulentnyi pogranichnyi sloi v sverkhzvukovykh potokakh gaza* (Turbulent Boundary Layer in Supersonic Gas Flows), Moscow: Nauka, 1982, 2nd ed.
6. Huang, P.G. and Bradshow, P., *AIAA J.*, 1995, vol. 33, no. 4, p. 624.
7. Townsend, A.A., *J. Fluid Mech.*, 1961, vol. 11, p. 97.
8. Lapin, Yu.V., *Turbulentnyi pogranichnyi sloi v sverkhzvukovykh potokakh gaza* (Turbulent Boundary Layer in Supersonic Gas Flows), Moscow: Nauka, 1970, 1st ed.
9. Fedyaevskii, K.K., Ginevskii, A.S., and Kolesnikov, A.V., *Raschet turbulentnogo pogranichnogo sloya neszhimaemoi zhidkosti* (Calculation of the Turbulent Boundary Layer of Incompressible Liquid), Leningrad: Sudostroenie, 1973.
10. Coles, D.E. and Hirst, E.A., Computation of Turbulent Boundary Layers, *AFOSR-IFP Stanford Conf.*, Palo Alto, CA: Stanford Univ., 1968, vol. 2.
11. Samuel, A.E. and Joubert, P.N., *J. Fluid Mech.*, 1974, vol. 66, no. 3, p. 481.
12. Andersen, P.S., Kays, W.M., and Moffat, R.J., *The Turbulent Boundary on a Porous Plate: An Experimental Study of the Fluid Mechanics for Adverse Free-Stream Pressure Gradients*, Thermosciences Division, Department of Mech. Eng., Stanford Univ., 1972, Report no. HMT-15.
13. Spalart, P.R. and Allmaras, S.R., *AIAA Pap.*, 1992, AIAA-92-0439.
14. Menter, F.R., *AIAA Pap.*, 1993, AIAA-93-2906.
15. Garbaruk, A.V., Lapin, Yu.V., and Strelets, M.Kh., *Teplofiz. Vys. Temp.*, 1998, vol. 36, no. 4, p. 607 (*High Temp.* (Engl. transl.), vol. 36, no. 4, p. 583).